Based on K. H. Rosen: Discrete Mathematics and its Applications.

Lecture 4: Predicates and quantifiers. Section 1.4

1 Predicates and quantifiers

We are going to introduce in this section a type of logic called **predicate logic**. A **predicate or propositional function** involves statements that contain variables, which may be true or false depending on those variables, for example we can denote the **statement** "x is greater than 3" by $P(x)$, where P denotes the predicate "is greater than 3" while x is the variable or the subject. In this particular example $P(4)$ is true and $P(2)$ is false. An statement can contain several variables:

Example 1. Let $Q(x, y)$ denote the statement "$x = y+3$." What are the truth values of the propositions $Q(1, 2)$ and $Q(3, 0)$?
Ans: $Q(1, 2)$ is false while $Q(3, 0)$ is true.

Definition 2. A statement of the form $P(x_1, x_2, \cdots, x_n)$ is the value of the propositional function P at the n-tuple $(x_1, x_2, \cdots, x_n)$, and P is also called an n-place predicate or a n-ary predicate.

1.1 Quantification

When the variables in a propositional function are assigned values, the resulting statement becomes a proposition with a certain truth value. The other way to create a proposition from a propositional function is quantification. Quantification expresses the extent to which a predicate is true over a range of elements. The area of logic that deals with predicates and quantifiers is called the **predicate calculus**.

Definition 3. The **universal quantification** of $P(x)$ is the statement "$P(x)$ for all values of x in the domain." The notation $\forall\, x\ P(x)$ denotes the universal quantification of $P(x)$. Here $\forall$ is called the universal quantifier. We read $\forall\, x\ P(x)$ as "for all x $P(x)$" or "for every x $P(x)$." An element for which $P(x)$ is false is called a counterexample of $\forall\, x\ P(x)$.

Definition 4. The **existential quantification** of $P(x)$ is the proposition "There exists an element x in the domain such that $P(x)$." We use the notation $\exists x\, P(x)$ for the existential quantification of $P(x)$. Here $\exists$ is called the existential quantifier.

Remark 5. Note that the domain where we are working must be specified.

Example 6. Let $Q(x)$ denote the statement "$x > 10$". What is the truth value of the quantifications $\exists\, x\, Q(x)$ and $\forall\, x\, Q(x)$, where the domain consists of all real numbers?
Ans: $\exists\, x\, Q(x)$ is true since, $Q(x)$ is true sometimes, for example $Q(11)$ is true. On the other hand $\forall\, x\, Q(x)$ is false because $Q(x)$ is false in some cases, like $Q(9)$ is false.

Example 7. What is the truth value of $\exists x\, P(x)$, where $P(x)$ is the statement "$x^2 > 10$" and the universe of discourse consists of the positive integers not exceeding 4? Ans: Since we have a finite list of elements in the domain, $\exists x\, P(x)$ is equivalent to

$$P(1) \vee P(2) \vee P(3) \vee P(4).$$

Because $P(4)$, which is the statement "$4^2 > 10$," is true, it follows that $\exists x\, P(x)$ is true.

We have now introduced universal and existential quantifiers. These are the most important quantifiers in mathematics and computer science. However, there is no limitation on the number of different quantifiers we can define, such as "there are exactly two" or "there are no more than three," . Of these other quantifiers, the one that is most often seen is the **uniqueness quantifier**, denoted by $\exists!$. The notation $\exists!\, P(x)$ states "There exists a unique x such that $P(x)$ is true."

Definition 8. When a quantifier is used on the variable x, we say that this occurrence of the variable is **bound**. An occurrence of a variable that is not bound by a quantifier or set equal to a particular value is said to be **free**. All the variables that occur in a propositional function must be bound or set equal to a particular value to turn it into a proposition.

Example 9. In the statement $\exists x\ (x + y = 1)$, the variable x is bound by the existential quantification $\exists x$, but the variable y is free because it is not bound by a quantifier and no value is assigned to this variable.

Definition 10. Statements involving predicates and quantifiers are logically equivalent if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions. We use the notation $S \equiv T$ to indicate that two statements S and T involving predicates and quantifiers are logically equivalent.

Example 11. We have that $\forall x\, (P(x) \wedge Q(x)$ and $\forall x\, P(x) \wedge \forall x\, Q(x)$ are logically equivalent. Indeed, for every point a in the domain $P(a) \wedge Q(a)$ is true if and only if $P(a)$ is true and $Q(a)$ is true.

Example 12. $\neg\forall x\, P(x) \equiv \exists x\, \neg P(x)$.
The proposition $\neg\forall x\, P(x)$ is true if and only if $\forall x\, P(x)$ is false. We note that $\forall x\, P(x)$ is false if and only if there is an element x in the domain for which $P(x)$ is false. This holds if and only if there is an element x in the domain for which $\neg\, P(x)$ is true. Finally, note that there is an element x in the domain for which $\neg\, P(x)$ is true if and only if $\exists x\, \neg P(x)$ is true.

Example 13. $\neg\exists x\, P(x) \equiv \forall x\, \neg P(x)$.
The proposition $\neg\exists x\, P(x)$ is true if and only if $\exists x\, P(x)$ is false. This happens if and only if there is not element x in the domain for which $P(x)$ is true. This holds if and only if for every element x in the domain $\neg\, P(x)$ is true or equivalently, the statement $\forall x\, \neg P(x)$.

Example 14. Express the statements "Some student in this class has visited Mexico" and "Every student in this class has visited either Canada or Mexico" using predicates and quantifiers.
Ans: Consider the domain of all persons and $S(x)$ the statement that says "x is a student of the class". We also introduce $M(x)$, which is the statement "x has visited Mexico" and $C(x)$, which is the statement "x has visited Canada". With this information we can write the statements using quantifiers as:

$$\exists x\,(S(x) \wedge M(x)) \qquad \text{and} \qquad \forall x\,(S(x) \rightarrow M(x) \vee C(x)).$$

Example 15. Consider these statements. The first two are called premises and the third is called the conclusion. The entire set is called an argument.
"All lions are fierce."
"Some lions do not drink coffee."
"Some fierce creatures do not drink coffee."
Ans: Let $P(x)$, $Q(x)$, and $R(x)$ be the statements "x is a lion" "x is fierce" and "x drinks coffee" respectively. Assuming that the domain consists of all creatures, express the statements in the argument using quantifiers:

$$\forall x(P(x) \rightarrow Q(x)) \qquad \exists x\,(P(x) \wedge \neg R(x)$$

and the conclusion will be:

$$\exists x\,(Q(x) \wedge \neg R(x).$$

Questions:

(1) Express each of these statements using predicates and quantifiers.
(a) The system mailbox can be accessed by everyone in the group if the file system is locked.
(b) The file system cannot be backed up if there is a user currently logged on.
(c) The diagnostic monitor tracks the status of all systems except the main console.
(d) Wheneverthereisanactivealert,allqueuedmessages are transmitted.